Worked Examples

Topic 1.1: Introduction and System Modelling

ENGM X304 – Applied Control Systems

Assoc. Prof. William Robertson   Dr Sean McGowan

October 3, 2025

Contents

1 Concept 1.1.1: Introduction

1.1 Example 1: Placeholder

This section will contain worked examples for the Introduction concept.

Problem: Example problem statement will be added here.

Solution: Example solution will be added here.

2 Concept 1.1.2: Concepts

2.1 Example 1: Placeholder

This section will contain worked examples for the Concepts concept.

Problem: Example problem statement will be added here.

Solution: Example solution will be added here.

3 Concept 1.1.3: State Space

3.1 Example 1: Placeholder

This section will contain worked examples for the State Space concept.

Problem: Example problem statement will be added here.

Solution: Example solution will be added here.

4 Concept 1.1.4: Modelling

4.1 Example 1: Free Falling Skydiver with Nonlinear Drag

Problem: A skydiver of mass $m = 80$ kg jumps from an aircraft. Model the vertical motion of the skydiver considering gravitational force and nonlinear air drag. The drag force is proportional to the square of velocity: $F_d = \frac{1}{2}\rho C_d A v^2$, where $\rho = 1.2$ kg/m^3 is air density, $C_d = 1.0$ is the drag coefficient, and $A = 0.5$ m^2 is the cross-sectional area.

Solution: Applying Newton's second law in the vertical direction (taking downward as positive):

$$m\frac{dv}{dt} = mg - \frac{1}{2}\rho C_d A v^2 \tag{1}$$

This can be rewritten as:

$$\frac{dv}{dt} = g - \frac{\rho C_d A}{2m} v^2 \tag{2}$$

Let $k = \frac{\rho C_d A}{2m}$, then:

$$\frac{dv}{dt} = g - kv^2 \tag{3}$$

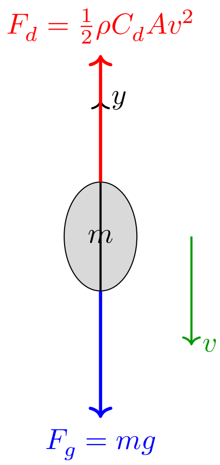

Figure 1: Free body diagram of a falling skydiver with forces acting on the body.

Substituting the given values:

$$k = \frac{1.2 \times 1.0 \times 0.5}{2 \times 80} = 0.00375 \text{ m}^{-1} \tag{4}$$

The terminal velocity occurs when $\frac{dv}{dt} = 0$:

$$v_{\text{terminal}} = \sqrt{\frac{g}{k}} = \sqrt{\frac{9.81}{0.00375}} \approx 51.1 \text{ m/s} \tag{5}$$

The state-space model is:

$$\frac{dv}{dt} = 9.81 - 0.00375v^2 \tag{6}$$

$$\frac{dy}{dt} = v \tag{7}$$

This is a nonlinear first-order differential equation in v and can be solved numerically or analytically using separation of variables.

4.2 Example 2: Flow Rate and Temperature in a Pipe

Problem: Consider a heated pipe system where fluid flows at rate q (m^3/s) through a pipe of volume $V = 0.1 \text{ m}^3$. The fluid enters at temperature T_{in} and is heated by a heater providing power P (W). Model the outlet temperature T considering heat input and convective flow. The fluid has density $\rho = 1000 \text{ kg/m}^3$ and specific heat capacity $c_p = 4200 \text{ J/(kg} \cdot \text{K)}$.

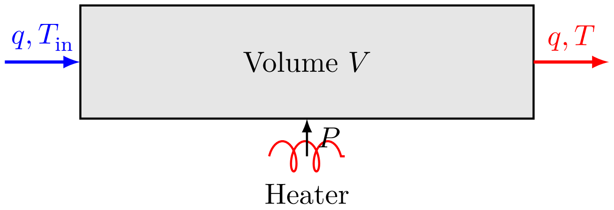

Figure 2: Schematic of a heated pipe system with flow rate and temperature dynamics.

Solution: Energy balance for the fluid in the pipe:

$$\frac{d}{dt}(\rho V c_p T) = \rho q c_p T_{\text{in}} - \rho q c_p T + P \tag{8}$$

Assuming constant volume and properties:

$$\rho V c_p \frac{dT}{dt} = \rho q c_p (T_{\text{in}} - T) + P \tag{9}$$

Dividing by $\rho V c_p$:

$$\frac{dT}{dt} = \frac{q}{V}(T_{\text{in}} - T) + \frac{P}{\rho V c_p} \tag{10}$$

Let $\tau = \frac{V}{q}$ be the residence time, then:

$$\frac{dT}{dt} = \frac{1}{\tau}(T_{\text{in}} - T) + \frac{P}{\rho V c_p} \tag{11}$$

For $q = 0.01$ m^3/s:

$$\tau = \frac{0.1}{0.01} = 10 \text{ s} \tag{12}$$

The state-space model is:

$$\frac{dT}{dt} = 0.1(T_{\text{in}} - T) + \frac{P}{420000} \tag{13}$$

This is a first-order linear system with two inputs (T_{in} and P) and one state (T). At steady state ($\frac{dT}{dt} = 0$):

$$T_{\text{ss}} = T_{\text{in}} + \frac{P}{\rho q c_p} \tag{14}$$

4.3 Example 3: Predator-Prey Population Dynamics (Foxes and Rabbits)

Problem: Model the population dynamics of rabbits (prey) and foxes (predators) in an ecosystem. Let $R(t)$ be the rabbit population and $F(t)$ be the fox population. Use the Lotka-Volterra model with the following assumptions:

- Rabbits reproduce at rate $\alpha = 0.1$ per month in the absence of predators
- Foxes prey on rabbits at rate $\beta = 0.02$ per month per rabbit
- Foxes die at rate $\gamma = 0.4$ per month without food
- Fox population grows at rate $\delta = 0.01$ per rabbit consumed

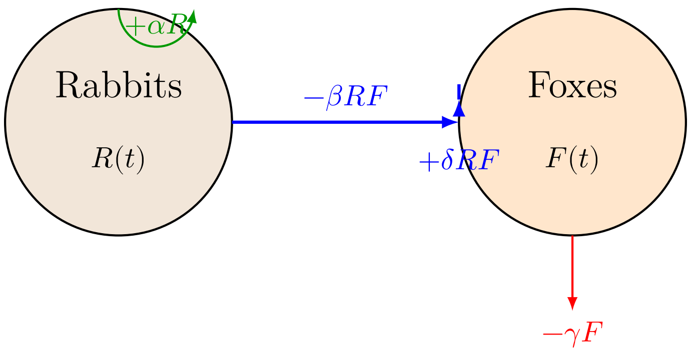

Figure 3: Population dynamics showing interaction between rabbits (prey) and foxes (predators).

Solution: The Lotka-Volterra equations are:

$$\frac{dR}{dt} = \alpha R - \beta RF \tag{15}$$

$$\frac{dF}{dt} = -\gamma F + \delta RF \tag{16}$$

Substituting the given parameters:

$$\frac{dR}{dt} = 0.1R - 0.02RF \tag{17}$$

$$\frac{dF}{dt} = -0.4F + 0.01RF \tag{18}$$

This is a nonlinear system with two state variables (R, F).
Equilibrium points occur when $\frac{dR}{dt} = \frac{dF}{dt} = 0$:
Equilibrium 1: $R = 0, F = 0$ (extinction)
Equilibrium 2: Setting both derivatives to zero:

$$0.1R - 0.02RF = 0 \quad \Rightarrow \quad R(0.1 - 0.02F) = 0 \tag{19}$$

$$-0.4F + 0.01RF = 0 \quad \Rightarrow \quad F(-0.4 + 0.01R) = 0 \tag{20}$$

For non-trivial solution:

$$0.1 - 0.02F = 0 \quad \Rightarrow \quad F^* = 5 \tag{21}$$

$$-0.4 + 0.01R = 0 \quad \Rightarrow \quad R^* = 40 \tag{22}$$

The equilibrium point is $(R^*, F^*) = (40, 5)$. This system exhibits periodic oscillations around this equilibrium, characteristic of predator-prey dynamics.

4.4 Example 4: Operational Amplifier Inverting Configuration

Problem: Model an operational amplifier in the inverting configuration with input resistance $R_1 = 10$ kΩ and feedback resistance $R_f = 100$ kΩ. Assume the op-amp has finite open-loop gain $A = 10^5$ and a single-pole frequency response with bandwidth $\omega_0 = 10$ rad/s. Derive the closed-loop transfer function.

Solution: For an ideal op-amp with finite gain A:

$$v_{\text{out}} = -Av_n \tag{23}$$

where v_n is the voltage at the inverting input (node voltage).

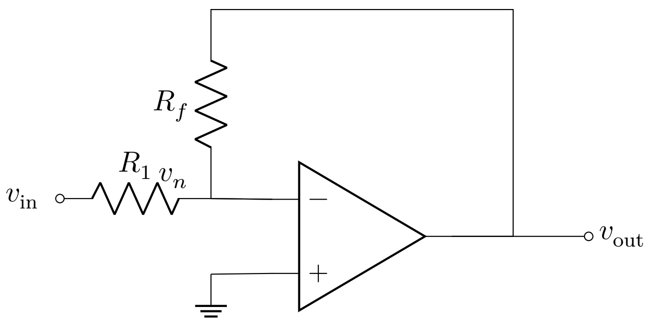

Figure 4: Inverting operational amplifier configuration with input and feedback resistors.

Applying Kirchhoff's current law at the inverting input node:

$$\frac{v_{\text{in}} - v_n}{R_1} + \frac{v_{\text{out}} - v_n}{R_f} = 0 \tag{24}$$

Substituting $v_{\text{out}} = -Av_n$:

$$\frac{v_{\text{in}} - v_n}{R_1} + \frac{-Av_n - v_n}{R_f} = 0 \tag{25}$$

Solving for v_n:

$$v_{\text{in}} - v_n + \frac{R_1}{R_f}(-Av_n - v_n) = 0 \tag{26}$$

$$v_{\text{in}} = v_n \left(1 + \frac{R_1}{R_f}(A+1)\right) \tag{27}$$

$$v_n = \frac{v_{\text{in}}}{1 + \frac{R_1}{R_f}(A+1)} \tag{28}$$

The closed-loop gain is:

$$\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{-Av_n}{v_{\text{in}}} = \frac{-A}{1 + \frac{R_1}{R_f}(A+1)} \tag{29}$$

For large A, this simplifies to:

$$\frac{v_{\text{out}}}{v_{\text{in}}} \approx \frac{-A}{\frac{R_1}{R_f}A} = -\frac{R_f}{R_1} \tag{30}$$

With $R_1 = 10$ kΩ and $R_f = 100$ kΩ:

$$\frac{v_{\text{out}}}{v_{\text{in}}} \approx -10 \tag{31}$$

For frequency response with single-pole dynamics, let $A(s) = \frac{A_0}{1+s/\omega_0}$:

$$\frac{v_{\text{out}}(s)}{v_{\text{in}}(s)} = \frac{-R_f/R_1}{1 + \frac{1+R_f/R_1}{A_0}(1 + s/\omega_0)} \tag{32}$$

This shows that the op-amp circuit acts as an inverting amplifier with gain determined primarily by the resistor ratio, with finite bandwidth effects from the op-amp's frequency response.